

# EDGE-REINFORCED RANDOM WALK, VERTEX-REINFORCED JUMP PROCESS AND THE SUPERSYMMETRIC HYPERBOLIC SIGMA MODEL

CHRISTOPHE SABOT AND PIERRE TARRES

**ABSTRACT.** Edge-reinforced random walk (ERRW), introduced by Coppersmith and Diaconis in 1986 [7], is a random process, which takes values in the vertex set of a graph  $G$ , and is more likely to cross edges it has visited before. We show that it can be represented in terms of a Vertex-reinforced jump process (VRJP) with independent gamma conductances: the VRJP was conceived by Werner and first studied by Davis and Volkov [9, 10], and is a continuous-time process favouring sites with more local time. We calculate, for any finite graph  $G$ , the limiting measure of the centred occupation time measure of VRJP, and interpret it as a supersymmetric hyperbolic sigma model in quantum field theory [15]. This enables us to deduce that VRJP and ERRW are strongly recurrent in any dimension for large reinforcement, using a localisation result of Disertori and Spencer [14].

## 1. INTRODUCTION

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let  $G = (V, E, \sim)$  be a nonoriented connected locally finite graph without loops. Let  $(a_e)_{e \in E}$  be a sequence of positive initial weights associated to each edge  $e \in E$ .

Let  $(X_n)_{n \in \mathbb{N}}$  be a random process that takes values in  $V$ , and let  $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$  be the filtration of its past. For any  $e \in E$ ,  $n \in \mathbb{N} \cup \{\infty\}$ , let

$$(1.1) \quad Z_n(e) = a_e + \sum_{k=1}^n \mathbb{1}_{\{X_{k-1}, X_k\} = e}$$

be the number of crosses of  $e$  up to time  $n$  plus the initial weight  $a_e$ .

Then  $(X_n)_{n \in \mathbb{N}}$  is called Edge Reinforced Random Walk (ERRW) with starting point  $i_0 \in V$  and weights  $(a_e)_{e \in E}$ , if  $X_0 = i_0$  and, for all  $n \in \mathbb{N}$ ,

$$(1.2) \quad \mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j \sim X_n\}} \frac{Z_n(\{X_n, j\})}{\sum_{k \sim X_n} Z_n(\{X_n, k\})}.$$

The Edge Reinforced Random Walk was introduced in 1986 by Diaconis [7]; on finite graphs it is a mixture of reversible Markov chains, and the mixing measure can be determined explicitly (the so-called Coppersmith-Diaconis measure, or "magic formula" [11], see also [16, 27]), which has applications in Bayesian statistics [13, 1, 2].

On infinite graphs, the research has focused so far on recurrence/transience criteria. On acyclic or directed graphs, the walk can be seen as a random walk in an *independent* random environment [25], and a recurrence/transience phase transition was first observed by Pemantle on trees [4, 17, 25]. In the case of infinite graphs with cycles,

---

2000 *Mathematics Subject Classification.* primary 60K37, 60K35, secondary 81T25, 81T60.

This work was partly supported by the ANR projects MEMEMO and MEMEMO2, and by a Leverhulme Prize.

recurrence criteria and asymptotic estimates were obtained by Merkl and Rolles on graphs of the form  $\mathbb{Z} \times G$ ,  $G$  finite graph, and on a certain two-dimensional graph [21, 22, 24, 23, 28], but recurrence on  $\mathbb{Z}^2$  was still unresolved.

Also, this original ERRW model [7] has triggered a number of similar models of self-organization and learning behaviour; see for instance Davis [8], Limic and Tarrès [19, 20], Pemantle [26], Sabot [29, 30], Tarrès [32, 33] and Tóth [34], with different perspectives on the topic.

Our first result relates the ERRW to the Vertex-Reinforced Jump Process (VRJP), conceived by Werner and studied by Davis and Volkov [9, 10], Collevecchio [5, 6] and Basdevant and Singh [3].

We call VRJP with weights  $(W_e)_{e \in E}$  a continuous-time process  $(Y_t)_{t \geq 0}$  on  $V$ , starting at time 0 at some vertex  $i_0 \in V$  and such that, if  $Y$  is at a vertex  $i \in V$  at time  $t$ , then, conditionally on  $(Y_s, s \leq t)$ , the process jumps to a neighbour  $j$  of  $i$  at rate  $W_{\{i,j\}} L_j(t)$ , where

$$L_j(t) := 1 + \int_0^t \mathbb{1}_{\{Y_s=j\}} ds.$$

The main results of the paper are the following. In Section 2, Theorem 1, we represent the ERRW in terms of a VRJP with independent gamma conductances. Section 3 is dedicated to showing, in Theorem 2, that the VRJP is a mixture of time-changed Markov jump processes, with a computation of the mixing law. In Section 6, we interpret that mixing law with the supersymmetric hyperbolic sigma model introduced by Disertori, Spencer and Zirnbauer in [15] and related to the Anderson model. We prove strong recurrence of VRJP and ERRW in any dimension for large reinforcement in Corollaries 1 and 2, using a localization result of Disertori and Spencer [14].

## 2. FROM ERRW TO VRJP.

It is convenient here to consider a time changed version of  $(Y_s)_{s \geq 0}$ : consider the positive continuous additive functional of  $(Y_s)_{s \geq 0}$

$$A(s) = \int_0^s \frac{1}{L_{Y_u}(u)} du = \sum_{x \in V} \log(L_x(s)),$$

and the time changed process

$$X_t = Y_{A^{-1}(t)}.$$

Let  $(T_i(t))_{i \in V}$  be the local time of the process  $(X_t)_{t \geq 0}$

$$T_x(t) = \int_0^t \mathbb{1}_{\{X_u=x\}} du.$$

**Lemma 1.** *The inverse functional  $A^{-1}$  is given by*

$$A^{-1}(t) = \int_0^t e^{T_{X_u}(u)} du = \sum_{i \in V} (e^{T_i(t)} - 1).$$

*The law of the process  $X_t$  is described by the following: conditioned on the past at time  $t$ , if the process  $X_t$  is at the position  $i$ , then it jumps to a neighbor  $j$  of  $i$  at rate*

$$W_{i,j} e^{T_i(t) + T_j(t)}.$$

*Proof.* First note that

$$(2.1) \quad T_x(A(s)) = \log(L_x(s)),$$

since

$$(T_x(A(s)))' = A'(s) \mathbb{1}_{\{X_{A(s)}=x\}} = \frac{1}{L_{Y_s(s)}} \mathbb{1}_{\{Y_s=x\}}.$$

Hence,

$$(A^{-1}(t))' = \frac{1}{A'(A^{-1}(t))} = L_{X_t}(A^{-1}(t)) = e^{T_{X_t}(t)},$$

which yields the expression for  $A^{-1}$ . It remains to prove the last assertion:

$$\begin{aligned} \mathbb{P}(X_{t+dt} = j | \mathcal{F}_t) &= \mathbb{P}(Y_{A^{-1}(t+dt)} = j | \mathcal{F}_t) \\ &= W_{X_t, j}(A^{-1})'(t) L_j(A^{-1}(t)) dt \\ &= W_{i, j} e^{T_{X_t}(t)} e^{T_j(t)} dt \end{aligned}$$

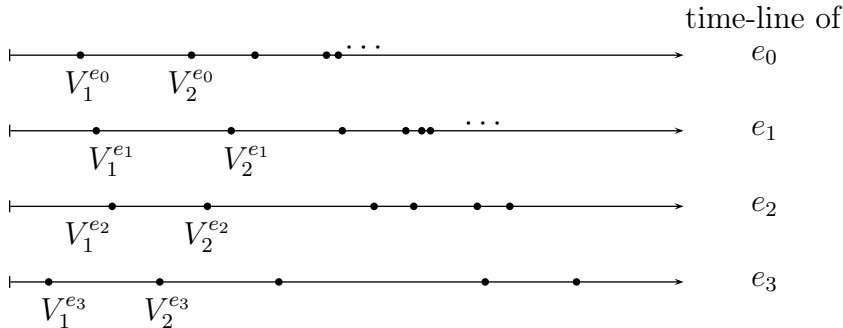
□

In order to relate ERRW to VRJP, let us first define the following process  $(\tilde{X}_t)_{t \in \mathbb{R}_+}$ , initially introduced by Rubin, Davis and Sellke [8, 31], which we call here continuous-time ERRW with weights  $(a_e)_{e \in E}$  and starting at  $\tilde{X}_0 := i_0$  at time 0.

- Define on each edge  $e \in E$  independent point processes (alarm times) as follows. Let  $(\tau_k^e)_{e \in E, k \in \mathbb{Z}_+}$  be independent exponential random variables with parameter 1 and define

$$V_k^e = \sum_{l=0}^{k-1} \frac{1}{a_e + l} \tau_l^e, \quad \forall k \in \mathbb{N}.$$

- Each edge  $e \in E$  has its own clock, denoted by  $\tilde{T}_e(t)$ , which only runs when the process  $(\tilde{X}_t)_{t \geq 0}$  is adjacent to  $e$ . This means that if  $e = \{i, j\}$ , then  $\tilde{T}_{\{i, j\}}(t) = \tilde{T}_i(t) + \tilde{T}_j(t)$ , where  $\tilde{T}_i(t)$  is the local time of the process  $\tilde{X}$  at vertex  $i$  and time  $t$ .
- When the clock of an edge  $e \in E$  rings, i.e. when  $\tilde{T}_e(t) = V_k^e$  for some  $k > 0$ , then  $\tilde{X}_t$  crosses it instantaneously (of course, this can happen only when  $\tilde{X}$  is adjacent to  $e$ ).



Let  $\tau_n$  be the  $n$ -th jump time of  $(\tilde{X}_t)_{t \geq 0}$ , with the convention that  $\tau_0 := 0$ .

**Lemma 2.** (Davis [8], Sellke [31]) Let  $(X_n)_{n \in \mathbb{N}}$  (resp.  $(\tilde{X}_t)_{t \geq 0}$ ) be an ERRW (resp. continuous-time ERRW) with weights  $(a_e)_{e \in E}$ , starting at some vertex  $i_0 \in V$ . Then  $(\tilde{X}_{\tau_n})_{n \geq 0}$  and  $(X_n)_{n \geq 0}$  have the same distribution.

*Proof.* The argument is based on the memoryless property of exponentials, and on the observation that, if  $A$  and  $B$  are two independent random variables of parameters  $a$  and  $b$ , then  $\mathbb{P}[A < B] = a/(a + b)$ .  $\square$

**Theorem 1.** *Let  $(\tilde{X}_t)_{t \geq 0}$  be a continuous-time ERRW with weights  $(a_e)_{e \in E}$ . Then there exists a sequence of independent random variables  $W_e \sim \text{Gamma}(a_e, 1)$ ,  $e \in E$ , such that, conditionally on  $(W_e)_{e \in E}$ ,  $(\tilde{X}_t)_{t \geq 0}$  has the same law as the time modification  $(X_t)_{t \geq 0}$  of the VRJP with weights  $(W_e)_{e \in E}$ .*

*In particular, the ERRW  $(X_n)_{n \geq 0}$  is equal in law to the discrete time process associated with a VRJP in random independent conductances  $W_e \sim \text{Gamma}(a_e, 1)$ .*

*Proof.* For any  $e \in E$ , define the simple birth process  $\{N_t^e, t \geq 0\}$  with initial population size  $a_e$ , by

$$N_t^e := a_e + \sup \{k \in \mathbb{N} \text{ s.t. } V_k^e \leq t\}.$$

This process is sometimes called the Yule process: by a result of D. Kendall [18], there exists  $W_e := \lim N_t^e e^{-t}$ , with distribution  $\text{Gamma}(a_e, 1)$ , such that, conditionally on  $W_e$ ,  $\{N_{f_{W_e}^e(t)}^e, t \geq 0\}$  is a Poisson point process with unit parameter, where

$$f_W(t) := \log(1 + t/W).$$

Let us now condition on  $(W_e)_{e \in E}$ :  $N_e$  increases between times  $t$  and  $t + dt$  with probability  $W_e e^t dt = (f_{W_e}^{-1})'(t) dt$ . A similar characterization of the timelines is also used in [33], Lemma 4.7. If  $\tilde{X}$  is at vertex  $x$  at time  $t$ , it jumps to a neighbour  $y$  of  $x$  at rate  $W_{x,y} e^{T_x(t) + T_y(t)}$ .  $\square$

### 3. THE MIXING MEASURE OF VRJP.

Next we study VRJP. Given fixed weights  $(W_e)_{e \in E}$ , we denote by  $(Y_t)_{t \geq 0}$  the VRJP and  $(X_t)_{t \geq 0}$  its time modification defined in the previous Section, starting at site  $X_0 := i_0$  at time 0 and  $(T_i(t))_{i \in V}$  its local time.

It is clear from the definition that the joint process  $\Theta_t = (X_t, (T_i(t))_{i \in V})$  is a time continuous Markov process on the state space  $V \times \mathbb{R}^V$  with generator  $\tilde{L}$  defined on  $C^\infty$  bounded functions by

$$\tilde{L}(f)(i, T) = \left( \frac{\partial}{\partial T_i} f \right)(i, T) + L(T)(f(\cdot, T))(i), \quad \forall (i, T) \in V \times \mathbb{R}_+^V,$$

where  $L(T)$  is the generator of the jump process on  $V$  at frozen  $T$  defined for  $g \in \mathbb{R}^V$ :

$$L(T)(g)(i) = \sum_{j \in V} W_{i,j} e^{T_i + T_j} (g(j) - g(i)), \quad \forall i \in V.$$

We denote by  $\mathbb{P}_{i_0, T}$  the law of the Markov process with generator  $\tilde{L}$  starting from the initial state  $(i_0, T)$ .

Note that the law of  $(X_t, T(t) - T)$  under  $\mathbb{P}_{i_0, T}$  is equal to the law of the process starting from  $(i_0, 0)$  with conductances

$$W_{i,j}^T = W_{i,j} e^{T_i + T_j}.$$

For simplicity, we let  $\mathbb{P}_i := \mathbb{P}_{i, 0}$ .

We show, in Proposition 1, that for finite graphs the centred occupation times converge a.s., and calculate the limiting measure in Theorem 2 i). In Theorem 2 ii)

we show that the VRJP  $(Y_s)_{s \geq 0}$  (as well as  $(X_t)_{t \geq 0}$ ) is a mixture of time-changed Markov jump processes.

This limiting measure can be interpreted as a supersymmetric hyperbolic sigma model. We are grateful to a few specialists of field theory for their advice: Denis Perrot who mentioned that the limit measure of VRJP could be related to the sigma model, and Krzysztof Gawedzki who pointed out reference [15], which actually mentions a possible link of their model with ERRW, suggested by Kozma, Heydenreich and Sznitman, cf [15] Section 1.5.

Note that when  $G$  is a tree, if the edges are for instance oriented towards the root, letting  $V_e = e^{U_{\tau} - U_{\sharp}}$ , the random variables  $(V_e)$  are independent and are distributed according to an inverse gaussian law. This was understood in previous works on VRJP [9, 10, 5, 6, 3].

Theorems 1 and 2 enable us to retrieve, in Section 5 the limiting measure of ERRWs, computed by Coppersmith and Diaconis in [7] (see also [16]), by integration over the random gamma conductances  $(W_e)_{e \in E}$ . This explains its renormalization constant, which had remained mysterious so far.

**Proposition 1.** *Suppose that  $G$  is finite and set  $N = |V|$ . For all  $i \in V$ , the following limits exist  $\mathbb{P}_{i_0}$  a.s.*

$$U_i = \lim_{t \rightarrow \infty} T_i(t) - \frac{t}{N}.$$

**Theorem 2.** *Suppose that  $G$  is finite and set  $N = |V|$ .*

i) *Under  $\mathbb{P}_{i_0}$ ,  $(U_i)_{i \in V}$  has the following density distribution on  $\mathcal{H}_0 = \{(u_i), \sum u_i = 0\}$*

$$(3.1) \quad \frac{1}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W, u)} \sqrt{D(W, u)},$$

where

$$H(W, u) = 2 \sum_{\{i, j\} \in E} W_{i, j} \sinh^2 \left( \frac{1}{2} (u_i - u_j) \right)$$

and  $D(W, u)$  is any diagonal minor of the  $N \times N$  matrix  $M(W, u)$  with coefficients

$$m_{i, j} = \begin{cases} W_{i, j} e^{u_i + u_j} & \text{if } i \neq j \\ -\sum_{k \in V} W_{i, k} e^{u_i + u_k} & \text{if } i = j \end{cases}$$

ii) *Let  $C$ , resp.  $D$ , be positive continuous additive functionals of  $X$ , resp.  $Y$ :*

$$C(t) = \sum_{i \in V} (e^{2T_i(t)} - 1), \quad D(s) = \sum_{i \in V} L_i^2(s) - 1$$

and let

$$Z_t = X_{C^{-1}(t)} (= Y_{D^{-1}(t)}).$$

*Then, conditionally on  $(U_i)_{i \in V}$ ,  $Z_t$  is a Markov jump process starting from  $i_0$ , with jump rate from  $i$  to  $j$*

$$\frac{1}{2} W_{i, j} e^{U_j - U_i}.$$

*In particular, the discrete time process associated with  $(Y_s)_{s \geq 0}$  is a mixture of reversible Markov chains with conductances  $W_{i, j} e^{U_i + U_j}$ .*

N.B.: 1) the density distribution in (3.1) is with respect to the Lebesgue measure on  $\mathcal{H}_0$  which is  $\prod_{i \in V \setminus \{j_0\}} du_i$  for any choice of  $j_0$  in  $V$ . We simply write  $du$  for any of the  $\prod_{i \in V \setminus \{j_0\}} du_i$ .

2) The diagonal minors of the matrix  $M(W, u)$  are all equal since the sum on any line or column of the coefficients of the matrix are null. By the matrix-tree theorem, if we let  $\mathcal{T}$  be the set of spanning trees of  $(V, E, \sim)$ , then  $D(W, u) = \sum_{T \in \mathcal{T}} \prod_{\{i,j\} \in T} W_{\{i,j\}} e^{u_i + u_j}$ .

**Remark 1.** *Remark that usually a result like ii) makes use of de Finetti's theorem: here, we provide a direct proof exploiting the explicit form of the density. In Section 5, we apply Theorem 1 and Theorem 2 i) ii) to give a new proof of Diaconis-Coppersmith formula including its de Finetti part.*

**Remark 2.** *The fact that (3.1) is a density is not at all obvious. Our argument is probabilistic: (3.1) is the law of the random variables  $(U_i)$ . It can also be explained directly as a consequence of supersymmetry, see (5.1) in [15]. The fact that the measure (3.1) normalizes at 1 is a fundamental property, which plays a crucial role in the localization and delocalization results of Disertori and Spencer [14, 15].*

**Remark 3.** *ii) implies that the VRJP  $(Y_s)$  is a mixture of Markov jump processes. More precisely, let  $(U_i)_{i \in V}$  be a random variable distributed according to (3.1) and, conditionally on  $U$ ,  $Z$  be the Markov jump process with jump rates from  $i$  to  $j$  given by  $\frac{1}{2} W_{i,j} e^{U_j - U_i}$ . Then the time changed process  $(Z_{B^{-1}}(s))_{s \geq 0}$  with*

$$B(t) = \sum_{i \in V} \sqrt{1 + l_i^Z(t)} - 1,$$

*where  $(l_i^Z(t))$  is the local time of  $Z$  at time  $t$ , has the law of the VRJP  $(Y_s)$  with conductances  $W$ .*

#### 4. PROOF OF THE PROPOSITION 1 AND THEOREM 2

**4.1. Proof of Proposition 1.** By a slight abuse of notation, we also use notation  $L(T)$  for the  $N \times N$  matrix  $M(W^T, T)$  of that operator in the canonical basis. Let  $\mathbb{I}$  be the  $N \times N$  matrix with coefficients equal to 1, i.e.  $\mathbb{I}_{i,j} = 1$  for all  $i, j \in V$ , and let  $I$  be the identity matrix.

Let us define, for all  $T \in \mathbb{R}^V$ ,

$$(4.1) \quad Q(T) := - \int_0^\infty \left( e^{uL(T)} - \frac{\mathbb{I}}{N} \right) du,$$

which exists since  $e^{uL(T)}$  converges towards  $\mathbb{I}/N$  at exponential rate.

Then  $Q(T)$  is a solution of the Poisson equation for the Markov Chain  $L(T)$ , namely

$$L(T)Q(T) = Q(T)L(T) = I - \frac{\mathbb{I}}{N}.$$

Observe that  $L(T)$  is symmetric, and thus  $Q(T)$  as well.

For all  $T \in \mathbb{R}^V$  and  $i, j \in V$ , let  $E_i^T(\tau_j)$  denote the expectation of the first hitting time of site  $j$  for the continuous-time process with generator  $L(T)$ . Then

$$Q(T)_{i,j} = \frac{1}{N} E_i^T(\tau_j) + Q(T)_{j,j}$$

by the strong Markov property applied to (4.1). As a consequence,  $Q(T)_{j,j}$  is nonpositive for all  $j$ , using  $\sum_{i \in V} Q(T)_{i,j} = 0$ .

Let us fix  $l \in V$ . We want to study the asymptotics of  $T_l(t) - t/N$  as  $t \rightarrow \infty$ :

$$\begin{aligned}
 T_l(t) - \frac{t}{N} &= \int_0^t \left( \mathbb{1}_{\{X_u=l\}} - \frac{1}{N} \right) du = \int_0^t (L(T(u))Q(T(u)))_{X_u,l} du \\
 &= \int_0^t \tilde{L}(Q(\cdot, \cdot)_l)(X_u, T(u)) du - \int_0^t \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} du \\
 (4.2) \quad &= Q(T(t))_{X_t,l} - Q(0)_{X_0,l} + M_l(t) - \int_0^t \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} du,
 \end{aligned}$$

where

$$M_l(t) := -Q(T(t))_{X_t,l} + Q(0)_{X_0,l} + \int_0^t \tilde{L}(Q(\cdot, \cdot)_l)(X_u, T(u)) du$$

is a martingale for all  $l$ . Recall that  $\tilde{L}$  is the generator of  $(X_t, T(t))$ .

The following lemma shows in particular the convergence of  $Q(T(t))_{k,l}$  for all  $k, l$ , as  $t$  goes to infinity. It is a purely deterministic statement, which does not depend on the trajectory of the process  $X_t$  (as long as it only performs finitely many jumps in a finite time interval), but only on the added local time in  $W^T$ .

**Lemma 3.** *For all  $k, l \in V$ ,  $Q(T(t))_{k,l}$  converges as  $t$  goes to infinity, and*

$$\int_0^\infty \left| \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} \right| du < \infty.$$

*Proof.* For all  $i, k, l \in V$ , let us compute  $\frac{\partial}{\partial T_i} Q(T)_{k,l}$ : by differentiation of the Poisson equation,

$$\frac{\partial}{\partial T_i} Q(T)_{k,l} = - \left( Q(T) \left( \frac{\partial}{\partial T_i} L \right) Q(T) \right)_{k,l}.$$

Now, for any real function  $f$  on  $V$ ,

$$\frac{\partial}{\partial T_i} Lf(k) = \begin{cases} \sum_{j \sim i} W_{i,j}^T (f(j) - f(i)) & \text{if } k = i \\ W_{i,k}^T (f(i) - f(k)) & \text{if } k \sim i, k \neq i \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\frac{\partial}{\partial T_i} Lf(k) = \sum_{j \sim i} W_{i,j}^T (f(j) - f(i)) (\mathbb{1}_{\{i=k\}} - \mathbb{1}_{\{j=k\}})$$

and, therefore,

$$\begin{aligned}
 \frac{\partial}{\partial T_i} Q(T)_{k,l} &= \sum_{j \sim i} W_{i,j}^T (Q(T)_{k,i} - Q(T)_{k,j}) (Q(T)_{i,l} - Q(T)_{j,l}) \\
 (4.3) \quad &= \sum_{j \sim i} W_{i,j}^T Q(T)_{k, \nabla_{i,j}} Q(T)_{\nabla_{i,j}, l} = \sum_{j \sim i} W_{i,j}^T Q(T)_{\nabla_{i,j}, k} Q(T)_{\nabla_{i,j}, l},
 \end{aligned}$$

where we use the notation  $f(\nabla_{i,j}) := f(j) - f(i)$  in the second equality, and the fact that  $Q(T)$  is symmetric in the third one.

In particular, for all  $l \in V$  and  $t \geq 0$ ,

$$(4.4) \quad \frac{d}{dt} Q(T(t))_{l,l} = \frac{\partial}{\partial T_{X_t}} Q(T(t))_{l,l} = \sum_{j \sim X_t} W_{X_t,j} (Q(T(t))_{\nabla_{X_t,j}, l})^2.$$

Now recall that  $Q(T(t))_{l,l}$  is nonpositive for all  $t \geq 0$ ; therefore it must converge, and

$$\int_0^\infty \sum_{j \sim X_t} W_{X_t,j} (Q(T(t))_{\nabla_{X_t,j,l}})^2 dt = (Q(T(\infty)) - Q(0))_{l,l} < \infty.$$

The convergence of  $Q(T(t))_{k,l}$  now follows from Cauchy-Schwarz inequality, using (4.3): for all  $t \geq s$ ,

$$\begin{aligned} |(Q(T(t)) - Q(T(s)))_{k,l}| &= \int_s^t \sum_{j \sim X_u} W_{X_u,j}^T Q(T(u))_{\nabla_{X_u,j,k}} Q(T(u))_{\nabla_{X_u,j,l}} du \\ &\leq \sqrt{(Q(T(t)) - Q(T(s)))_{k,k}} \sqrt{(Q(T(t)) - Q(T(s)))_{l,l}}, \end{aligned}$$

thus  $Q(T(t))_{k,l}$  is Cauchy sequence, which converges as  $t$  goes to infinity. Now, using again Cauchy-Schwarz inequality,

$$\begin{aligned} &\int_0^\infty \left| \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} \right| du \\ &= \int_0^\infty \left| \sum_{j \sim X_u} W_{X_u,j}^T Q(T(u))_{\nabla_{X_u,j,X_u}} Q(T(u))_{\nabla_{X_u,j,l}} \right| du \\ &\leq \sqrt{\sum_{k \in V} (Q(T(\infty)) - Q(T(0)))_{k,k}} \sqrt{(Q(T(\infty)) - Q(T(0)))_{l,l}}, \end{aligned}$$

which enables us to conclude.  $\square$

Next, we show that  $(M_l(t))_{t \geq 0}$  converges, which will complete the proof: indeed, this implies that the size of the jumps in that martingale goes to 0 a.s., and therefore, by (4.2), that  $Q(T(t))_{X_t,l}$  must converge as well, again by (4.2).

Let us compute the quadratic variation of the martingale  $(M_l(t))_{t \geq 0}$  at time  $t$ :

$$\begin{aligned} &\left( \frac{d}{d\varepsilon} \mathbb{E} ((M_l(T(t+\varepsilon)) - M_l(t))^2 | \mathcal{F}_t) \right)_{\varepsilon=0} \\ &= \left( \frac{d}{d\varepsilon} \mathbb{E} ((Q(T(t+\varepsilon))_{X_{t+\varepsilon},l} - Q(T(t))_{X_t,l})^2 | \mathcal{F}_t) \right)_{\varepsilon=0} \\ &= R(T(t))_{X_t,l} \end{aligned}$$

where, for all  $(i, l, T) \in V \times V \times \mathbb{R}^V$ , we let

$$R(T)_{i,l} := \tilde{L}(Q^2(\cdot)_{.,l})(i, T) - 2Q(T)_{i,l} \tilde{L}(Q(\cdot)_{.,l})(i, T);$$

here  $Q^2(T)$  denotes the matrix with coefficients  $(Q(T)_{i,j})^2$ , rather than  $Q(T)$  composed with itself. But

$$\begin{aligned} \tilde{L}(Q^2(\cdot)_{.,l})(i, T) &= 2(Q(T))_{i,l} \left( \frac{\partial}{\partial T_i} Q(T) \right)_{i,l} + (L(T)Q^2(T)_{.,l}(i))_{i,l} \\ Q(T)_{i,l} \tilde{L}(Q(\cdot)_{.,l})(i, T) &= (Q(T))_{i,l} \left( \frac{\partial}{\partial T_i} Q(T) \right)_{i,l} + Q(T)_{i,l} (L(T)Q(T)_{.,l}(i))_{i,l}, \end{aligned}$$



so that

$$\begin{aligned}
R(T)_{i,l} &= L(T)(Q^2(T)_{.,l})_{i,l} - 2Q(T)_{i,l}(L(T)Q(T)_{.,l})_{i,l} \\
&= \sum_{j \sim i} W_{i,j}^T ((Q(T)_{j,l})^2 - (Q(T)_{i,l})^2) - 2Q(T)_{i,l} \sum_{j \sim i} W_{i,j}^T (Q(T)_{j,l} - Q(T)_{i,l}) \\
&= \sum_{j \sim i} W_{i,j}^T (Q(T)_{\nabla_{i,j},l})^2 = \frac{\partial}{\partial T_i} Q(T)_{l,l},
\end{aligned}$$

using (4.3) in the last equality. Thus

$$\langle M_l, M_l \rangle_\infty = \int_0^\infty \frac{d}{du} Q(u)_{l,l} du = Q(T(\infty))_{l,l} - Q(0)_{l,l} \leq -Q(0)_{l,l} < \infty.$$

Therefore  $(M_l(t))_{t \geq 0}$  is a martingale bounded in  $L^2$ , which converges a.s.

**Remark 4.** Once we know that  $T_i(t) - t/N$  converges, then  $T_i(\infty) = \infty$  for all  $i \in V$ , hence  $Q(T(\infty))_{l,l} = 0$ , and the last inequality is in fact an equality, i.e.  $\langle M_l, M_l \rangle_\infty = -Q(0)_{l,l}$ .

**4.2. Proof of Theorem 2 i).** We consider, for  $i_0 \in V$ ,  $T \in \mathbb{R}^V$ ,  $\lambda \in \mathcal{H}_0$

$$(4.5) \quad \Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{u_{i_0}} e^{i \langle \lambda, u \rangle} \phi(W^T, u) du,$$

where

$$(4.6) \quad \phi(W^T, u) = e^{-H(W^T, u)} \sqrt{D(W^T, u)},$$

and  $W_{i,j}^T = W_{i,j} e^{T_i + T_j}$ . We will prove that

$$\frac{1}{\sqrt{2\pi}^{N-1}} \Psi(i_0, T, \lambda) = \mathbb{E}_{i_0, T} (e^{i \langle \lambda, U \rangle}),$$

for all  $i_0 \in V$ ,  $T \in \mathbb{R}^V$ .

**Lemma 4.** The function  $\Psi$  is solution of the Feynman-Kac equation

$$i\lambda_{i_0} \Psi(i_0, T, \lambda) + (\tilde{L}\Psi)(i_0, T, \lambda) = 0.$$

*Proof.* Let  $\bar{T}_i = T_i - \frac{1}{N} \sum_{j \in V} T_j$ . With the change of variables  $\tilde{u}_i = u_i + \bar{T}_i$ , we obtain

$$(4.7) \quad \Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{\tilde{u}_{i_0} - \bar{T}_{i_0}} e^{i \langle \lambda, \tilde{u} - \bar{T} \rangle} \phi(W^T, \tilde{u} - \bar{T}) d\tilde{u}$$

Remark now that  $H(W^T, \tilde{u} - \bar{T}) = H(W^T, \tilde{u} - T)$  since  $H(W^T, u)$  only depends on the differences  $u_i - u_j$ . We observe that the coefficients of the matrix  $M(W^T, u)$  only contain terms of the form  $W_{i,j} e^{u_i + T_i + u_j + T_j}$ , hence

$$\sqrt{D(W^T, \tilde{u} - \bar{T})} = e^{\frac{N-1}{N} \sum_j T_j} \sqrt{D(W, \tilde{u})}.$$

Finally,  $\langle \lambda, \bar{T} \rangle = \langle \lambda, T \rangle$  since  $\lambda \in \mathcal{H}_0$ . This implies that

$$(4.8) \quad \Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{\sum_j T_j} e^{\tilde{u}_{i_0} - T_{i_0}} e^{i \langle \lambda, \tilde{u} - T \rangle} e^{-H(W^T, \tilde{u} - T)} \sqrt{D(W, \tilde{u})} d\tilde{u}.$$

We have

$$\begin{aligned}
& \frac{\partial}{\partial T_{i_0}} H(W^T, \tilde{u} - T) \\
&= \frac{\partial}{\partial T_{i_0}} \left( 2 \sum_{\{i,j\} \in E} W_{i,j} e^{T_i + T_j} \sinh^2 \left( \frac{1}{2} (\tilde{u}_i - \tilde{u}_j - T_i + T_j) \right) \right) \\
&= 2 \sum_{j \sim i_0} W_{i_0,j} e^{T_{i_0} + T_j} \left( \sinh^2 \left( \frac{1}{2} (\tilde{u}_{i_0} - \tilde{u}_j - T_{i_0} + T_j) \right) - \frac{1}{2} \sinh(\tilde{u}_{i_0} - \tilde{u}_j - T_{i_0} + T_j) \right) \\
&= \sum_{j \sim i_0} W_{i_0,j} e^{T_{i_0} + T_j} (e^{-\tilde{u}_{i_0} + \tilde{u}_j + T_{i_0} - T_j} - 1) \\
&= e^{-(\tilde{u}_{i_0} - T_{i_0})} L(T)(e^{\tilde{u} - T})(i_0).
\end{aligned}$$

Hence,

$$\begin{aligned}
& -\frac{\partial}{\partial T_{i_0}} \Psi(i_0, T, \lambda) \\
&= \int_{\mathcal{H}_0} (i\lambda_{i_0} e^{\tilde{u}_{i_0} - T_{i_0}} + L(T)(e^{\tilde{u} - T})(i_0)) e^{\sum_j T_j} e^{i\langle \lambda, \tilde{u} - T \rangle} e^{-H(W^T, \tilde{u} - T)} \sqrt{D(W, \tilde{u})} d\tilde{u} \\
&= i\lambda_{i_0} \Psi(i_0, T, \lambda) + (L(T)\Psi)(i_0, T, \lambda).
\end{aligned}$$

This gives

$$(\tilde{L}\Psi)(i_0, T, \lambda) = -i\lambda_{i_0} \Psi(i_0, T, \lambda).$$

□

Since  $\Psi$  is a solution of the Feynman-Kac equation we deduce that for all  $t > 0$ ,  $i_0 \in V$ ,  $\lambda \in \mathcal{H}_0$ ,  $T \in \mathbb{R}^V$ ,

$$\Psi(i_0, T, \lambda) = \mathbb{E}_{i_0, T} \left( e^{i\langle \lambda, \bar{T}(t) \rangle} \Psi(X_t, T(t), \lambda) \right),$$

where we recall that  $\bar{T}_i(t) = T_i(t) - t/N$ . Let us now prove that  $\Psi(X_t, T(t), \lambda)$  is dominated and that  $\mathbb{P}_{i_0}$  a.s.

$$(4.9) \quad \lim_{t \rightarrow \infty} \Psi(X_t, T(t), \lambda) = \sqrt{2\pi}^{N-1}.$$

By the matrix-tree theorem, we have, denoting by  $\mathcal{T}$  the set of spanning trees of  $G$ , and using again notation  $\phi$  in (4.6),

$$\begin{aligned}
(4.10) \quad e^{u_{i_0}} \phi(W^T, u) &= e^{u_{i_0}} e^{-H(W^T, u)} \sqrt{\sum_{\Lambda \in \mathcal{T}} \prod_{\{i,j\} \in \Lambda} W_{i,j}^T e^{u_i + u_j}} \\
&\leq e^{N \max_{i \in V} |u_i|} e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^T (u_i - u_j)^2} \sqrt{D(W^T, 0)} \\
&\leq \left( \sum_{i \in V} e^{Nu_i} + e^{-Nu_i} \right) e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^T (u_i - u_j)^2} \sqrt{D(W^T, 0)}
\end{aligned}$$

This is a gaussian integrand: for any real  $a$  and  $i_0 \in V$ ,

$$\begin{aligned} & \int_{\mathcal{H}_0} e^{au_{i_0}} e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^T (u_i - u_j)^2} \sqrt{D(W^T, 0)} du \\ &= e^{-\frac{1}{2} a^2 Q(T)_{i_0, i_0}} \int e^{-\frac{1}{2} \langle U - aQ(T)_{i_0, \cdot}, L(T)(U - aQ(T)_{i_0, \cdot}) \rangle} \sqrt{D(W^T, 0)} du \\ &= e^{-\frac{1}{2} a^2 Q(T)_{i_0, i_0}} (2\pi)^{(N-1)/2}. \end{aligned}$$

where  $Q(T)$  is defined at the beginning of Section 4.1. Therefore for all  $i_0 \in V$ ,  $(T_i) \in \mathbb{R}^V$

$$|\Psi(i_0, T, \lambda)| \leq 2 \sum_{i \in V} (2\pi)^{(N-1)/2} e^{-\frac{1}{2} N^2 Q(T)_{i,i}},$$

Using (4.4),  $Q(T(t))_{i,i}$  increases in  $t$ , hence

$$|\Psi(X_t, T(t), \lambda)| \leq 2 \sum_{i \in V} (2\pi)^{(N-1)/2} e^{-\frac{1}{2} N^2 Q(0)_{i,i}},$$

for all  $t \geq 0$ . Let us prove now (4.9). We have

$$\begin{aligned} & \Psi(X_t, T(t), \lambda) \\ &= \int e^{i \langle \lambda, u \rangle} e^{u_{X_t}} e^{-2 \sum_{\{i,j\} \in E} W_{i,j}^{T(t)} \sinh^2(\frac{1}{2}(u_i - u_j))} \sqrt{D(W^{T(t)}, u)} du \\ &= \int e^{i \langle \lambda, u \rangle} e^{u_{X_t}} e^{-2 \sum_{\{i,j\} \in E} e^{2t/N} W_{i,j}^{\bar{T}(t)} \sinh^2(\frac{1}{2}(u_i - u_j))} \sqrt{D(W^{\bar{T}(t)}, u)} e^{(N-1)t/N} du. \end{aligned}$$

Changing to variables  $\tilde{u}_i = e^{t/N} u_i$ , we deduce that  $\Psi(X_t, T(t), \lambda)$  equals

$$\int e^{i \langle \lambda, e^{-t/N} \tilde{u} \rangle} e^{e^{-t/N} \tilde{u}_{X_t}} e^{-2 \sum_{\{i,j\} \in E} W_{i,j}^{\bar{T}(t)} e^{2t/N} \sinh^2(\frac{1}{2} e^{-t/N} (\tilde{u}_i - \tilde{u}_j))} \sqrt{D(W^{\bar{T}(t)}, e^{-t/N} \tilde{u})} d\tilde{u}.$$

Since  $\lim_{t \rightarrow \infty} \bar{T}_i(t) = U_i$ , the integrand converges pointwise to the Gaussian integrand

$$e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^U (\tilde{u}_i - \tilde{u}_j)^2} \sqrt{D(W^U, 0)},$$

whose integral is  $\sqrt{2\pi}^{N-1}$ . Consider  $\bar{U}_i = \sup_{t \geq 0} \bar{T}_i(t)$  and  $\underline{U}_i = \inf_{t \geq 0} \bar{T}_i(t)$ . Proceeding as in (4.10) the integrand is dominated for all  $t$  by

$$\begin{aligned} & e^{Ne^{-t/N} \max_{i \in V} |\tilde{u}_i|} e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^{\bar{T}(t)} (\tilde{u}_i - \tilde{u}_j)^2} \sqrt{D(W^{\bar{T}(t)}, 0)} \\ & \leq \left( \sum_{i \in V} e^{N\tilde{u}_i} + e^{-N\tilde{u}_i} \right) e^{-\frac{1}{2} \sum_{\{i,j\} \in V} W_{i,j}^{\underline{U}} (\tilde{u}_i - \tilde{u}_j)^2} \sqrt{D(W^{\underline{U}}, 0)}. \end{aligned}$$

which is integrable, which yields (4.9) by dominated convergence.

**4.3. Proof of Theorem 2 ii).** The same change of variables as in (4.8), applied to  $T_i = \log \lambda_i$ , implies that, for any  $j_0 \in V$  and  $(\lambda_i)_{i \in V}$  positive reals,

$$\frac{\prod_{i \in V} \lambda_i}{\sqrt{2\pi}^{N-1}} e^{u_{j_0} - \log(\lambda_{j_0})} e^{-\frac{1}{2} \sum_{\{i,j\} \in E} W_{i,j} \lambda_i \lambda_j \left( e^{\frac{1}{2}(u_j - u_i)} \sqrt{\frac{\lambda_i}{\lambda_j}} - e^{\frac{1}{2}(u_j - u_i)} \sqrt{\frac{\lambda_j}{\lambda_i}} \right)^2} \sqrt{D(W, u)}$$

is the density of a probability measure, which we call  $\nu^{\lambda, j_0}$  (using that (3.1) defines a probability measure). Remark that this density can be rewritten as

$$\frac{\prod_{i \in V} \lambda_i}{\sqrt{2\pi}^{N-1}} e^{u_{j_0} - \log(\lambda_{j_0})} e^{-\frac{1}{2} \sum_i \sum_{j \sim i} W_{i,j} (\lambda_i^2 e^{u_j - u_i} - \lambda_i \lambda_j)} \sqrt{D(W, u)}$$

Let  $(U_i)$  be a random variable distributed according to (3.1), and, conditionally on  $U$ , let  $(Z_t)$  be the Markov jump process starting at  $i_0$ , and with jump rates from  $i$  to  $j$

$$\frac{1}{2}W_{i,j}e^{U_j-U_i}.$$

Let  $(\mathcal{F}_t^Z)_{t \geq 0}$  be the filtration generated by  $Z$ , and let  $E_i^U$  be the law of the process  $Z$  starting at  $i$ , conditionally on  $U$ .

We denote by  $(l_i(t))_{i \in V}$  the vector of local times of the process  $Z$  at time  $t$ , and consider the positive continuous additive functional of  $Z$

$$B(t) = \int_0^t \frac{1}{2} \frac{1}{\sqrt{1 + l_{Z_u}(u)}} du = \sum_{i \in V} \left( \sqrt{1 + l_i(t)} - 1 \right),$$

and the time changed process

$$\tilde{Y}_s = Z_{B^{-1}(s)}.$$

Let us first prove that the law of  $U$  conditioned on  $\mathcal{F}_t^Z$  is

$$(4.11) \quad \mathcal{L}(U|\mathcal{F}_t^Z) = \nu^{\lambda(t), Z_t},$$

where  $\lambda_i(t) = \sqrt{1 + l_i(t)}$ . Indeed, let  $t > 0$ : if  $\tau_1, \dots, \tau_{K(t)}$  denote the jumping times of the Markov process  $Z_t$  up to time  $t$ , then for any positive test function,

$$\begin{aligned} E_{i_0}^U \left( \psi(\tau_1, \dots, \tau_{K(t)}, Z_{\tau_1}, \dots, Z_{\tau_{K(t)}}) \right) = \\ \sum_{k=0}^{\infty} \sum_{i_1, \dots, i_k} \left( \prod_{l=0}^{k-1} \frac{1}{2} W_{i_l, i_{l+1}} \right) \int_{[0, t]^k} \psi((t_j), (i_j)) e^{U_{i_k} - U_{i_0}} e^{-\frac{1}{2} \sum_{l=0}^{k-1} \left( \sum_{j \sim i_l} W_{i_l, j} e^{U_j - U_{i_l}} \right) (t_{l+1} - t_l)} dt_1 \dots dt_k \end{aligned}$$

with the convention  $t_{k+1} = t$ . Hence, for any test function  $G$ ,

$$\mathbb{E} \left( G(U) | \mathcal{F}_t^Z \right) = \frac{\int_{\mathcal{H}_0} G(u) e^{u Z_t} e^{-H(W, u) - \frac{1}{2} \sum_{i \in V} \left( \sum_{j \sim i} W_{i, j} e^{u_j - u_i} \right) l_i(t)} \sqrt{D(W, u)} du}{\int_{\mathcal{H}_0} e^{u Z_t} e^{-H(W, u) - \frac{1}{2} \sum_{i \in V} \left( \sum_{j \sim i} W_{i, j} e^{u_j - u_i} \right) l_i(t)} \sqrt{D(W, u)} du}.$$

Using that we can write  $H(W, u) = \frac{1}{2} \sum_{i \in V} \sum_{j \sim i} (e^{u_j - u_i} - 1)$ , and introducing adequate constants in the numerator and denominator we have

$$\begin{aligned} \mathbb{E} \left( G(U) | \mathcal{F}_t^Z \right) \\ = \frac{\sqrt{2\pi}^{-(N-1)} \int_{\mathcal{H}_0} G(u) \left( \prod \lambda_i \right) e^{u Z_t - \log \lambda_{Z_t}} e^{-\frac{1}{2} \sum_i \sum_{j \sim i} W_{i, j} (\lambda_i(t)^2 e^{u_j - u_i} - \lambda_i(t) \lambda_j(t))} \sqrt{D(W, u)} du}{\sqrt{2\pi}^{-(N-1)} \int_{\mathcal{H}_0} \left( \prod \lambda_i \right) e^{u Z_t - \log \lambda_{Z_t}} e^{-\frac{1}{2} \sum_i \sum_{j \sim i} W_{i, j} (\lambda_i(t)^2 e^{u_j - u_i} - \lambda_i(t) \lambda_j(t))} \sqrt{D(W, u)} du} \end{aligned}$$

(recall that  $\lambda_i(t) = \sqrt{1 + l_i(t)}$ ). The denominator is 1 since it is the integral of the density of  $\nu^{\lambda(t), Z_t}$ . This proves (4.11).

Subsequently, by (4.11), conditioned on  $(\mathcal{F}_t^Z)$ , if the process  $Z$  is at  $i$  at time  $t$ , then it jumps to a neighbour  $j$  of  $i$  with rate

$$\frac{1}{2} W_{i, j} \mathbb{E}^{\nu^{\lambda(t), i}} (e^{U_j - U_i}) = \frac{1}{2} W_{i, j} \frac{\lambda_j(t)}{\lambda_i(t)}.$$

In order to conclude, we now compute the corresponding rate for  $\tilde{Y}$ : by definition,

$$B'(t) = \frac{1}{2} \frac{1}{\sqrt{1 + l_{Z_t}(t)}}.$$

Therefore, similarly as in the proof of Lemma 1,

$$\begin{aligned}\mathbb{P}\left(\tilde{Y}_{s+ds} = j | \mathcal{F}_t^Z\right) &= \mathbb{P}\left(Z_{B^{-1}(s+ds)} = j | \mathcal{F}_t^Z\right) \\ &= \frac{1}{2} W_{Y_s, j} \frac{1}{B'(B^{-1}(s))} \frac{\lambda_j(B^{-1}(s))}{\lambda_{Y_s}(B^{-1}(s))} ds \\ &= W_{Y_s, j} \lambda_j(B^{-1}(s)) ds.\end{aligned}$$

Let  $(\tilde{l}_i(s))$  be the local time of the process  $\tilde{Y}$ . Then

$$(\tilde{l}_i(B(t)))' = B'(t) \mathbb{1}_{\{\tilde{Y}_{B(s)}=i\}} = \frac{1}{2} (1 + l_i(t))^{-\frac{1}{2}} \mathbb{1}_{\{Z_t=i\}}.$$

This implies

$$(4.12) \quad \tilde{l}_i(B(t)) = \sqrt{1 + l_i(t)} - 1$$

and

$$\mathbb{P}\left(\tilde{Y}_{s+ds} = j | \mathcal{F}_t^Z\right) = W_{\tilde{Y}_s, j} (1 + \tilde{l}_j(s)) ds$$

This means that the annealed law of  $\tilde{Y}$  is the law of a VRJP with conductances  $(W_{i,j})$  (this is the content of remark 3).

Therefore, the process defined, for all  $t \geq 0$ , by  $\tilde{Y}_{A^{-1}(t)} = Z_{(A \circ B)^{-1}(t)}$ , is equal in law to  $(X_t)_{t \geq 0}$ ; let us denote by  $T$  its local time, and show that  $T_i(t) - t/N$  converges to  $U_i$  as  $t \rightarrow \infty$ , which will complete the proof.

First note, using (2.1) and (4.12), that, for all  $i \in V$ ,

$$T_i((A \circ B)(t)) = \log(\tilde{l}_i(B(t)) + 1) = \log(1 + l_i(t))/2.$$

On the other hand, conditionally on  $U$ , the Markov Chain  $Z$  has invariant measure  $(Ce^{2U_i})_{i \in V}$ ,  $C := (\sum_{i \in V} e^{2U_i})^{-1}$ , so that  $l_i(t)/(Ce^{2U_i}t)$  converges to 1 as  $t \rightarrow \infty$ , for all  $i \in V$ .

Therefore, for all  $i \in V$ ,

$$T_i(t) - T_{i_0}(t) = \frac{1}{2} \log \left( \frac{1 + l_i((A \circ B)^{-1}(t))}{1 + l_{i_0}((A \circ B)^{-1}(t))} \right),$$

which converges towards  $U_i - U_{i_0}$  as  $t \rightarrow \infty$ , which enables us to conclude.

## 5. BACK TO DIACONIS-COPPERSMITH FORMULA

It follows from de Finetti's theorem for Markov chains [12] that the law of the ERRW is a mixture of reversible Markov chains; its mixing measure was explicitly described by Coppersmith and Diaconis ([7], see also [16, 27]).

Theorems 1 and 2 enable us to retrieve this so-called Coppersmith-Diaconis formula, including its de Finetti part: they imply that the ERRW  $(X_n)_{n \in \mathbb{N}}$  follows the annealed law of a reversible Markov chain in a random conductance network  $x_{i,j} = W_{i,j} e^{U_i + U_j}$  where  $W_e \sim \text{Gamma}(a_e, 1)$ ,  $e \in E$ , are independent random variables and, conditioned on  $W$ , the random variables  $(U_i)$  are distributed according to the law (3.1).

Let us compute the law it induces on the random variables  $(x_e)$ . The random variable  $(x_e)$  is only significant up to a scaling factor, hence we consider a 0-homogeneous

bounded measurable test function  $\phi$ ; by Theorem 2,

$$\begin{aligned} & \mathbb{E}(\phi((x_e))) \\ &= \frac{1}{\sqrt{2\pi}^{N-1}} \int_{\mathbb{R}_+^E \times \mathcal{H}_0} \phi(x) \left( \prod_{e \in E} \frac{1}{\Gamma(a_e)} W_e^{a_e} e^{-W_e} \right) e^{u_{i_0}} \sqrt{D(W, u)} e^{-H(W, u)} \frac{dW}{W} du \end{aligned}$$

where we write  $\frac{dW}{W} = \prod_{e \in E} \frac{dW_e}{W_e}$ . Changing to coordinates  $\bar{u}_i = u_i - u_{i_0}$  yields

$$C(a) \int_{\mathbb{R}_+^E \times \mathbb{R}^{V \setminus \{i_0\}}} \phi(x) \left( \prod_{e \in E} W_e^{a_e} e^{-W_e} \right) e^{-\sum_{i \neq i_0} \bar{u}_i} \sqrt{D(W, \bar{u})} e^{-H(W, \bar{u})} \frac{dW}{W} d\bar{u}$$

with  $d\bar{u} = \prod_{i \neq i_0} d\bar{u}_i$  and  $C(a) = \frac{1}{\sqrt{2\pi}^{N-1}} \left( \prod_{e \in E} \frac{1}{\Gamma(a_e)} \right)$ . But

$$-\sum_{e \in E} W_e - H(W, \bar{u}) = -\frac{1}{2} \sum_{\{i, j\} \in E} W_{i, j} e^{\bar{u}_i + \bar{u}_j} (e^{-2\bar{u}_j} + e^{-2\bar{u}_i}).$$

The change of variables

$$((x_{i, j} = W_{i, j} e^{\bar{u}_i + \bar{u}_j})_{\{i, j\} \in E}, (v_i = e^{-2\bar{u}_i})_{i \in V \setminus \{i_0\}}),$$

with  $v_{i_0} = 1$  implies

$$-\sum_{e \in E} W_e - H(W, \bar{u}) = -\frac{1}{2} \sum_{i \in V} v_i x_i,$$

where  $x_i = \sum_{j \sim i} x_{i, j}$ , and  $\mathbb{E}(\phi((x_e)))$  is equal to the integral

$$C'(a) \int \phi(x) \left( \prod_{e \in E} x_e^{a_e} \right) \left( \prod_{i \in V} v_i^{(a_i+1)/2} \right) v_{i_0}^{-\frac{1}{2}} \sqrt{D(x)} e^{-\frac{1}{2} \sum_{i \in V} v_i x_i} \left( \prod_{e \in E} \frac{dx_e}{x_e} \right) \left( \prod_{i \neq i_0} \frac{dv_i}{v_i} \right),$$

with  $a_i = \sum_{j \sim i} a_{i, j}$ ,  $D(x)$  determinant of any diagonal minor of the  $N \times N$  matrix

$$m_{i, j} = \begin{cases} x_{i, j} & \text{if } i \neq j \\ -\sum_{k \sim i} x_{i, k} & \text{if } i = j \end{cases}$$

and

$$C'(a) = \frac{2^{-N+1}}{\sqrt{2\pi}^{N-1}} \left( \prod_{e \in E} \frac{1}{\Gamma(a_e)} \right).$$

Let  $e_0$  be a fixed edge: we normalize the conductance to be 1 at  $e_0$  by changing to variables

$$\left( \left( y_e = \frac{x_e}{x_{e_0}} \right)_{e \neq e_0}, (z_i = x_{e_0} v_i)_{i \in V} \right),$$

with  $y_{e_0} = 1$ . Now, observe that

$$\left( \prod_{e \in E} \frac{dx_e}{x_e} \right) \left( \prod_{i \neq i_0} \frac{dv_i}{v_i} \right) = \left( \prod_{e \in E, e \neq e_0} \frac{dy_e}{y_e} \right) \left( \prod_{i \in V} \frac{dz_i}{z_i} \right).$$

We deduce that  $\mathbb{E}(\phi((x_e)))$  equals the integral

$$C(a) \int_{\mathbb{R}_+^V \times \mathbb{R}_+^{E \setminus \{e_0\}}} \phi(y) \left( \prod_{e \in E} y_e^{a_e} \right) \left( \prod_{i \in V} z_i^{a_i/2} \right) z_{i_0}^{-\frac{1}{2}} \sqrt{D(y)} e^{-\frac{1}{2} \sum_{i \in V} z_i y_i} \left( \frac{dy}{y} \right) \left( \frac{dz}{z} \right),$$

with  $\frac{dy}{y} = \prod_{e \neq e_0} \frac{dy_e}{y_e}$  and  $\frac{dz}{z} = \prod_{i \in V} \frac{dz_i}{z_i}$ . Therefore, integrating over the variables  $z_i$

$$\mathbb{E}(\phi((x_e))) = C''(a) \int_{\mathbb{R}_+^{E \setminus \{e_0\}}} \phi(y) y_{i_0}^{\frac{1}{2}} \left( \frac{\prod_{e \in E} y_e^{a_e}}{\prod_{i \in V} y_i^{(a_i+1)/2}} \right) \sqrt{D(y)} \left( \frac{dy}{y} \right),$$

where

$$C''(a) = \frac{2^{1-N-\sum_{e \in E} a_e} \Gamma(a_{i_0}/2) \prod_{i \neq i_0} \Gamma((a_i+1)/2)}{\pi^{(N-1)/2} \prod_{e \in E} \Gamma(a_e)}$$

which is Diaconis-Coppersmith formula: the extra term  $(|E| - 1)!$  in [16, 13] arises from the normalization of  $(x_e)_{e \in E}$  on the simplex  $\Delta = \{\sum x_e = 1\}$  (see Section 2.2 [13]).

## 6. THE SUPERSYMMETRIC HYPERBOLIC SIGMA MODEL

We first relate VRJP to the supersymmetric hyperbolic sigma model studied in Disertori, Spencer and Zirnbauer [15, 14]. For notational purposes, we restrict our attention to the  $d$ -dimensional lattice, that is, our graph is  $\mathbb{Z}^d$  with  $x \sim y$  if  $|x - y|_1 = 1$ . We denote by  $E$  the set of edges  $E = \{\{i, j\}, i \sim j\}$ . For a subset  $\Lambda \subseteq \mathbb{Z}^d$  we denote by  $E_\Lambda$  the set of edges with both extremities in  $\Lambda$ .

We start by a description of the measures defined in [15, 14]. Let  $V \subseteq \mathbb{Z}^d$  be a connected finite subset containing 0. Let  $\beta_{i,j}, i, j \in V, i \sim j$  be some positive weights on the edges, and  $\varepsilon = (\varepsilon_i)_{i \in V}$  be a vector of non-negative reals,  $\varepsilon \neq 0$ . Let  $\mu_V^{\varepsilon, \beta}$  be a generalization of the measure studied in [14] (see (1.1)-(1.7) in that paper), namely

$$\begin{aligned} d\mu_V^{\varepsilon, \beta}(t) &:= \left( \prod_{j \in V} \frac{dt_j}{\sqrt{2\pi}} \right) e^{-\sum_{k \in V} t_k} e^{-F_V^\beta(\nabla t)} e^{-M_V^\varepsilon(t)} \sqrt{\det A_V^{\varepsilon, \beta}} \\ &= \left( \prod_{j \in V} \frac{dt_j}{\sqrt{2\pi}} \right) e^{-F_V^\beta(\nabla t)} e^{-M_V^\varepsilon(t)} \sqrt{\det D_V^{\varepsilon, \beta}} \end{aligned}$$

where  $A_V^{\varepsilon, \beta} = A^{\varepsilon, \beta}$  and  $D_V^{\varepsilon, \beta} = D^{\varepsilon, \beta}$  are defined by, for all  $i, j \in V$ , by

$$A_{ij}^{\varepsilon, \beta} = e^{t_i} D_{ij}^{\varepsilon, \beta} e^{t_j} = \begin{cases} 0 & |i - j| > 1 \\ -\beta_{ij} e^{t_i + t_j} & |i - j| = 1 \\ \sum_{l \sim i, l \in V} \beta_{il} e^{t_i + t_l} + \sum_{i \in V} \varepsilon_i e^{t_i} & i = j \end{cases}$$

$$F_V^\beta(\nabla t) := \sum_{\{i, j\} \in E_V} \beta_{ij} (\cosh(t_i - t_j) - 1)$$

$$M_V^\varepsilon(t) := \sum_{i \in V} \varepsilon_i (\cosh t_i - 1).$$

The fact that  $\mu_V^{\varepsilon, \beta}$  is a probability measure can be seen as a consequence of supersymmetry (see (5.1) in [15]). This is also a consequence of Theorem 2 i), cf later.

The measure  $\mu_V^{\varepsilon, \beta}$  is directly related to the measure (3.1) defined in theorem 2 as follows. Let add an extra point  $\delta$  to  $V$ ,  $\tilde{V} = V \cup \{\delta\}$ , and extra edges  $\{i, \delta\}$  connecting any site  $i \in V$  to  $\delta$ , i.e.  $\tilde{E}_V = E_V \cup \cup_{i \in V} \{i, \delta\}$ . Consider the VRJP on  $\tilde{V}$  starting at  $\delta$  and with conductances  $W_{i,j} = \beta_{i,j}$ , if  $i \sim j$  in  $V$ , and  $W_{i,\delta} = \varepsilon_i$ . Let us again use notation  $(U_i)_{i \in \tilde{V}}$  for the limiting centred occupation times of VRJP on  $\tilde{V}$  starting at  $\delta$ ,

and consider the change of variables, from  $\mathcal{H}_0$  into  $\mathbb{R}^V$ , which maps  $u_i$  to  $t_i := u_i - u_\delta$ . Then, by Theorem 2, for any test function  $\phi$ ,

$$\begin{aligned} \mathbb{E}_\delta^W(\phi(U - U_\delta)) &= \frac{1}{(2\pi)^{|V|/2}} \int_{\mathcal{H}_0} \phi(u - u_\delta) e^{u_\delta} e^{-H(W, u)} \sqrt{D(W, u)} du \\ &= \frac{1}{(2\pi)^{|V|/2}} \int_{\mathcal{H}_0} \phi(t) e^{-\sum_{i \in V} t_i} e^{-H(W, t)} \sqrt{D(W, t)} \left( \prod_{i \neq \delta} dt_i \right) \\ &= \mu_V^{\varepsilon, \beta}(\phi(t)), \end{aligned}$$

which means that  $U - U_\delta$  is distributed according to  $\mu_V^{\varepsilon, \beta}$ . Indeed, let  $\iota$  be the canonical injection  $\mathbb{R}^V \rightarrow \mathbb{R}^{\tilde{V}}$ ; then  $A_V^\varepsilon$  is the restriction to  $V \times V$  of the matrix  $M(W, \iota(t))$  (which is on  $\tilde{V} \times \tilde{V}$ ) (so that  $\det A_V^\varepsilon = D(W, \iota(t))$ ), and  $F_V(\nabla t) + M_V^\varepsilon(t) = H(W, \iota(t))$ .

We will be interested in the VRJP on finite subsets of  $\mathbb{Z}^d$  starting at 0. In order to apply directly results of [14] we consider the VRJP on  $\mathbb{Z}^d$  with an extra point  $\delta$  uniquely connected to 0 and with  $W_{0, \delta} = \epsilon_0 = 1$ ,  $W_{i, j} = \beta_{i, j}$ ,  $i \sim j$  in  $\mathbb{Z}^d$ . Clearly, the trace on  $\mathbb{Z}^d$  of the VRJP starting from  $\delta$  has the law of the VRJP on  $\mathbb{Z}^d$  starting from 0. When  $V$  contains 0 the limiting occupation time  $U_i - U_\delta$  of the VRJP on  $\tilde{V} = V \cup \{\delta\}$  starting at  $\delta$  is then distributed according to  $d\mu_V^{\delta_0, \beta}$ , where  $\delta_0$  is the Dirac at 0.

Set, for all  $\beta > 0$ ,

$$I_\beta := \sqrt{\beta} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-\beta(\cosh t - 1)},$$

which is strictly increasing in  $\beta$ . Let  $\beta_c^d$  be defined as the unique solution to the equation

$$I_{\beta_c^d} e^{\beta_c^d(2d-2)} (2d-1) = 1$$

for all  $d > 1$ ,  $\beta_c^d := \infty$  if  $d = 1$ .

If the parameters  $\beta_e$  are constant equal to  $\beta$  over all edges  $e$ , then Theorem 2 in [14] readily implies that VRJP over any graph  $\mathbb{Z}^d$  is recurrent for  $\beta < \beta_c^d$  (i.e. for large reinforcement).

**Theorem 3** (Disertori and Spencer [14], Theorem 2). *Suppose that  $\beta_e = \beta$  for all  $e$ . Then there exists a constant  $C_0 := 2d/(2d-1) > 0$  such that, for all  $\Lambda \subseteq \mathbb{Z}^d$  finite connected containing 0,  $x \in \Lambda$ ,  $0 < \beta < \beta_c^d$ ,*

$$\mu_\Lambda^{\eta\delta_0, \beta}(e^{t_x/2}) \leq C_0 I_\eta [I_\beta e^{\beta(2d-2)} (2d-1)]^{|x|}.$$

**Corollary 1.** *For  $0 < \beta < \beta_c^d$ , let  $(Y_n)$  be the discrete time process associated with the VRJP on  $\mathbb{Z}^d$  starting from 0 with constant conductance  $\beta$ . Then  $(Y_n)$  is a mixture of reversible strongly recurrent Markov chains.*

*Proof.* We prove this for the VRJP on  $\mathbb{Z}^d$  with an extra point  $\delta$  connected to 0 only, and conductances  $W_{x, y} = \beta$  and  $W_{0, \delta} = 1$ , which is clearly stronger. On a finite connected subset  $V \subseteq \mathbb{Z}^d$  containing 0, we know from Theorem 2 that  $(Y_n)_{n \in \mathbb{N}}$ , the discrete-time process associated with  $(Y_s)_{s \geq 0}$ , is a mixture of reversible Markov chains with conductances  $c_{x, y} = \beta e^{t_x + t_y}$ , where  $(t_x)_{x \in V}$  has law  $\mu_V^{\delta_0, \beta}$ .



Now Theorem 3 implies that  $\mu_V^{\delta_0, \beta}((c_e/c_{\delta, 0})^{1/4})$  decreases exponentially with the distance from  $e$  to 0: indeed, by Cauchy-Schwarz inequality,

$$\begin{aligned} \mu_V^{\delta_0, \beta}((c_{x,y}/c_{\delta, 0})^{1/4}) &\leq \left[ \mu_V^{\delta_0, \beta}(e^{t_x/2}) \mu_V^{\delta_0, \beta}(e^{(t_y-t_0)/2}) \right]^{1/2} \\ &\leq C \left[ \mu_V^{\delta_0, \beta}(e^{t_x/2}) \mu_V^{\delta_0, \beta}(e^{\frac{1}{2}(\cosh(t_0)-1)e^{t_y/2}}) \right]^{1/2} \\ &\leq 2C \left[ \mu_V^{\delta_0, \beta}(e^{t_x/2}) \mu_V^{\delta_0/2, \beta}(e^{t_y/2}) \right]^{1/2} \end{aligned}$$

for some  $C > 0$  such that  $|z| \leq 4 \log C + \cosh(z) - 1$ . This implies that there exists constants  $c_1 > 0$ ,  $c_2 > 0$ , such that  $\mu_V^{\delta_0, \beta}((c_{x,y}/c_{\delta, 0})^{1/4}) \leq e^{-c_1|x|} \leq e^{-c_2|x|}$ . Following the proof of lemma 5.1 of [22] it implies that  $(Y_n)$  is a mixture of strongly recurrent Markov chains.  $\square$

By Theorems 1 and 2, the ERRW with constant initial weights  $a > 0$  corresponds to the case where  $(\beta_e)_{e \in E}$  are independent random variables with  $\text{Gamma}(a, 1)$  distribution for some parameter  $a > 0$ : we can infer a similar localization and recurrence result for  $a$  small enough. This requires an extension of Theorem 3 for random gamma weights  $(\beta_e)_{e \in E}$ : we propose one in the following Proposition 2, in the same line of proof as in [14].

**Proposition 2.** *Let  $a$  be a positive real, and assume that the conductances  $(\beta_e)_{e \in E}$  are i.i.d. with law  $\text{Gamma}(a, 1)$ . Denote by  $\mathbb{E}$  the expectation with respect to the random variables  $(\beta_e)_{e \in E}$ . For all  $d \geq 1$ , there exists  $a_c^d > 0$ ,  $\delta \in (0, 1)$ , such that if  $a < a_c^d$ , there exists  $C_0 > 0$  depending only on  $a$  and  $d$  such that for all  $x \sim y$ ,*

$$\mathbb{E} \left( \mu_\Lambda^{\eta \delta_0, \beta}(e^{t_x/2}) \right) \leq C_0 I_\eta \delta^{|x|}, \quad \mathbb{E} \left( (\beta_{x,y})^{\frac{1}{4}} \mu_\Lambda^{\eta \delta_0, \beta}(e^{t_x/2}) \right) \leq C_0 I_\eta \delta^{|x|}$$

*independently on the finite connected subset  $\Lambda$  containing 0 and  $x, y$ .*

**Corollary 2.** *The ERRW on  $\mathbb{Z}^d$  starting at 0 with constant initial weight  $a > 0$  is a mixture of strongly recurrent Markov chains for  $a < a_c^d$  (where  $a_c^d$  is defined in Proposition 2).*

**Remark 5.** *Corollary 2 also holds on any graph of bounded degree, and for possibly non-constant weights  $(a_e)_{e \in E}$  with  $a_e < a_c$  for some  $a_c > 0$ . Indeed, the proof of Proposition 2 also holds for independent (not necessarily i.i.d.) conductances  $(\beta_e)$  with  $\mathbb{E}(\sqrt{\beta_e}(\log(1 + \beta_e^{-1})))$  sufficiently small, when the graph is of bounded degree.*

*Proof.* (Proposition 2) The strategy is to follow the proof of [14], Theorem 2, and to truncate the random variables  $\beta_e$  at adequate positions. For convenience we provide a self-contained proof but the only new input compared to [14], Theorem 2, lies in the threshold argument (6.5–6.7). Let us define, for any  $\Lambda \subseteq \mathbb{Z}^d$ ,  $(\epsilon_i)_{i \in \Lambda} \in \mathbb{R}_+^\Lambda$

$$d\nu_\Lambda^{\epsilon, \beta}(t) := \left( \prod_{i \in \Lambda} \frac{dt_i}{\sqrt{2\pi}} \right) e^{-F_\Lambda^\beta(\nabla t)} e^{-M_\Lambda^\epsilon(t)},$$

which is not a probability measure in general.

We fix now a finite connected subset  $\Lambda \subseteq \mathbb{Z}^d$  containing 0, and  $x$ . Let  $\Gamma_x$  be the set of non-intersecting paths in  $\Lambda$  from 0 to  $x$ . For notational purposes, any element  $\gamma$  in  $\Gamma_x$  is defined here as the set of non-oriented edges in the path. We let  $\Lambda_\gamma$  and  $\Lambda_\gamma^c$  be respectively the set of vertices in the path and its complement. We say that an

edge  $e$  is adjacent to the path  $\gamma$  if  $e$  is not in  $\gamma$  and has one adjacent vertex in  $\gamma$ , i.e. if  $e = \{i, j\}$  with  $i \in \Lambda_\gamma, j \notin \Lambda_\gamma$ ; we write  $e \sim \gamma$ .

We first proceed similarly to (3.1)-(3.4) in [14], Lemma 2. Let  $\mathcal{T}_\Lambda$  be the set of spanning trees of  $\Lambda$ . By the matrix-tree theorem

$$\det(A_\Lambda^{\eta\delta_0, \beta}) = \eta e^{t_0} \sum_{T \in \mathcal{T}_\Lambda} \prod_{\{i, j\} \in T} \beta_{\{i, j\}} e^{t_i + t_j}.$$

In a spanning tree  $T$  there is a unique path between 0 and  $x \in \Lambda$ . Decomposing this sum depending on this path we deduce

$$\det(A_\Lambda^{\eta\delta_0, \beta}) = \eta e^{t_0} \sum_{\gamma \in \Gamma_x} \left( \prod_{\{i, j\} \in \gamma} \beta_{\{i, j\}} e^{t_i + t_j} \right) \sum_{T' \in \mathcal{T}_\Lambda^\gamma} \prod_{\{i, j\} \in T'} \beta_{\{i, j\}} e^{t_i + t_j}$$

where  $\mathcal{T}_\Lambda^\gamma$  is the set of subsets  $T' \subseteq E_\Lambda \setminus \gamma$  such that  $\gamma \cup T'$  is a spanning tree. By the matrix-tree theorem, we have

$$(6.1) \quad \sum_{T' \in \mathcal{T}_\Lambda^\gamma} \prod_{\{i, j\} \in T'} \beta_{\{i, j\}} e^{t_i + t_j} = \det(A_{\Lambda_\gamma^c}^{\epsilon, \beta})$$

where  $(\epsilon_i)_{i \in \Lambda_\gamma^c}$  is the vector defined by

$$\epsilon_i := \sum_{k \in \Lambda_\gamma, k \sim i} \beta_{\{i, k\}} e^{t_k}, \quad \forall i \in \Lambda_\gamma^c$$

It follows that

$$(6.2) \quad \det D_\Lambda^{\eta\delta_0, \beta} = \eta e^{-t_x} \sum_{\gamma \in \Gamma_x} \left( \prod_{e \in \gamma} \beta_e \right) \det D_{\Lambda_\gamma^c}^{\epsilon, \beta}.$$

Let us define, similarly as in (2.12) and (2.14) in [14], for  $t_\gamma = t|_{\Lambda_\gamma}$  the restriction of  $t$  to the vertices on the path  $\gamma$ ,

$$(6.3) \quad \begin{aligned} Z_{\Lambda_\gamma^c}^{\gamma, \beta}(t_\gamma) &:= \nu_{\Lambda_\gamma^c}^{\eta\delta_0, \beta} \left( \sqrt{\det D_{\Lambda_\gamma^c}^{\epsilon, \beta}} e^{-F_{\partial\gamma}^\beta(\nabla t)} \right) \\ F_{\partial\gamma}^\beta(\nabla t) &:= \sum_{k \in \Lambda_\gamma, j \in \Lambda_\gamma^c, k \sim j} \beta_{kj} (\cosh(t_j - t_k) - 1). \end{aligned}$$

Now

$$(6.4) \quad \begin{aligned} \mu_\Lambda^{\eta\delta_0, \beta}(e^{t_x/2}) &= \nu_\Lambda^{\eta\delta_0, \beta} \left( \sqrt{\det D_\Lambda^{\eta\delta_0, \beta}} e^{t_x} \right) = \sqrt{\eta} \nu_\Lambda^{\eta\delta_0, \beta} \left( \sqrt{\sum_{\gamma \in \Gamma_x} \prod_{e \in \gamma} \beta_e \det D_{\Lambda_\gamma^c}^{\epsilon, \beta}} \right) \\ &\leq \sqrt{\eta} \sum_{\gamma \in \Gamma_x} \left( \prod_{e \in \gamma} \sqrt{\beta_e} \right) \nu_{\Lambda_\gamma}^{\eta\delta_0, \beta} \left( Z_{\Lambda_\gamma^c}^{\gamma, \beta}(t_\gamma) \right), \end{aligned}$$

using (6.2) in the second equality and, in the inequality, that for all  $\gamma \in \Gamma_x$ ,

$$d\nu_\Lambda^{\eta\delta_0, \beta}(t) = d\nu_{\Lambda_\gamma}^{\eta\delta_0, \beta}(t) d\nu_{\Lambda_\gamma^c}^{\eta\delta_0, \beta}(t) e^{-F_{\partial\gamma}^\beta(\nabla t)}.$$

The new argument compared to theorem 3 which allows to handle the case of random parameters  $\beta$  is the following truncation. Given  $\gamma \in \Gamma_x$ , let  $(\tilde{\beta}_e)$  be the set of

random variables defined by

$$(6.5) \quad \tilde{\beta}_e = \begin{cases} \min(\beta_e, 1), & \text{if } e \sim \gamma, \\ \beta_e, & \text{otherwise.} \end{cases}$$

First note that, trivially,

$$(6.6) \quad e^{-F_{\partial\gamma}^\beta(\nabla t)} \leq e^{-F_{\partial\gamma}^{\tilde{\beta}}(\nabla t)}.$$

On the other hand, identity (6.1) implies that

$$(6.7) \quad \det(D_{\Lambda_\gamma^c}^{\varepsilon, \beta}) \leq \det(D_{\Lambda_\gamma^c}^{\tilde{\varepsilon}, \tilde{\beta}}) \left( \prod_{e \sim \gamma} \max(\beta_e, 1) \right),$$

where  $(\tilde{\varepsilon}_i)_{i \in \Lambda_\gamma^c}$  is the vector defined by

$$\tilde{\varepsilon}_i := \sum_{k \in \Lambda_\gamma, i \sim k} \tilde{\beta}_{\{i, k\}} e^{t_k}, \quad \forall i \in \Lambda_\gamma^c$$

(In the last argument we used that, for any  $\{i, j\}$  adjacent to  $\gamma$ ,  $\beta_{i,j} = \tilde{\beta}_{i,j} \max(1, \beta_{i,j})$ ). Therefore

$$(6.8) \quad Z_{\Lambda_\gamma^c}^{\gamma, \beta}(t_\gamma) \leq Z_{\Lambda_\gamma^c}^{\gamma, \tilde{\beta}}(t_\gamma) \prod_{e \sim \gamma} \sqrt{\max(\beta_e, 1)}$$

with  $Z_{\Lambda_\gamma^c}^{\gamma, \tilde{\beta}}(t_\gamma)$  defined as in (6.3) with  $\varepsilon, \beta$  replaced by  $\tilde{\varepsilon}, \tilde{\beta}$ . Hence we can replace  $\beta$  by  $\tilde{\beta}$  at the cost of the term  $\prod_{e \sim \gamma} \sqrt{\max(\beta_e, 1)}$ .

The following lemma, which adapts Lemma 3 [14], provides an upper bound of  $Z_{\Lambda_\gamma^c}^{\gamma, \tilde{\beta}}(t_\gamma)$ .

**Lemma 5.** *For any configuration of  $t_\gamma = t_{|\Lambda_\gamma}$ ,  $Z_{\Lambda_\gamma^c}^{\gamma, \tilde{\beta}}(t_\gamma) \leq e^{\sum_{e \sim \gamma} \tilde{\beta}_e}$ .*

*Proof.* We have

$$Z_{\Lambda_\gamma^c}^{\gamma, \tilde{\beta}}(t_\gamma) = \int \left( \frac{\prod_{j \in \Lambda_\gamma^c} dt_j}{\sqrt{2\pi}} \right) e^{-F_{\Lambda_\gamma^c}^{\tilde{\beta}}(\nabla t) - F_{\partial\gamma}^{\tilde{\beta}}(\nabla t)} \sqrt{\det(D_{\Lambda_\gamma^c}^{\tilde{\varepsilon}, \tilde{\beta}})}$$

Let  $t^* = \max\{t_k, k \in \Lambda_\gamma\}$ . We perform the following translation in the variables  $t_j \rightarrow t_j + t^*$  for  $j \in \Lambda_\gamma^c$ : in the previous integral the term  $F_{\Lambda_\gamma^c}^{\tilde{\beta}}(\nabla t)$  does not change, the term  $F_{\partial\gamma}^{\tilde{\beta}}(\nabla t)$  becomes

$$\sum_{k \in \Lambda_\gamma, j \in \Lambda_\gamma^c, k \sim j} \tilde{\beta}_{kj} (\cosh(t_j + t^* - t_k) - 1),$$

and the term  $\det(D_{\Lambda_\gamma^c}^{\tilde{\varepsilon}, \tilde{\beta}})$  is replaced by  $\det(D_{\Lambda_\gamma^c}^{e^{-t^*} \tilde{\varepsilon}, \tilde{\beta}})$ . Since  $t^* - t_k \geq 0$  we have

$$\cosh(t_j + t^* - t_k) - 1 \geq e^{t_k - t^*} (\cosh(t_j) - 1) + (e^{t_k - t^*} - 1).$$

This implies that

$$\sum_{k \in \Lambda_\gamma, j \in \Lambda_\gamma^c, k \sim j} \tilde{\beta}_{kj} (\cosh(t_j + t^* - t_k) - 1) \geq M_{\Lambda_\gamma^c}^{e^{-t^*} \tilde{\varepsilon}}(t) + \sum_{k \in \Lambda_\gamma, j \in \Lambda_\gamma^c, k \sim j} \tilde{\beta}_{kj} (e^{t_k - t^*} - 1),$$

and

$$Z_{\Lambda_\gamma^c}^{\gamma, \tilde{\beta}}(t_\gamma) \leq e^{\sum_{k \in \Lambda_\gamma, j \in \Lambda_\gamma^c, k \sim j} \tilde{\beta}_{kj} (1 - e^{t_k - t^*})} \mu_{\Lambda_\gamma^c}^{e^{-t^*} \tilde{\varepsilon}, \tilde{\beta}}(1) \leq e^{\sum_{e \sim \gamma} \tilde{\beta}_e},$$

using that  $\mu_{\Lambda_\gamma}^{e^{-t^*} \varepsilon, \tilde{\beta}}$  is a probability measure.  $\square$

Combining (6.4), (6.8), Lemma 5, and integration on the variables  $(\nabla t_e)_{e \in \gamma}$ , we obtain that

$$\mu_\Lambda^{\eta_{\delta_0, \beta}}(e^{t_x/2}) \leq I_\eta \sum_{\gamma \in \Gamma_x} \left( \prod_{e \sim \gamma} \sqrt{\max(\beta_e, 1)} e^{\min(\beta_e, 1)} \right) \left( \prod_{e \in \gamma} I_{\beta_e} \right)$$

We set

$$\hat{I}_a = \mathbb{E}(I_\beta) = \frac{\Gamma(a + \frac{1}{2})}{\Gamma(a)} \int_{-\infty}^{\infty} \cosh(t)^{-a - \frac{1}{2}} dt,$$

and

$$\hat{J}_a = \mathbb{E}(\max(\beta, 1) e^{\min(\beta, 1)}).$$

where  $\beta$  is a gamma random variable with parameter  $a$ . Clearly,  $\hat{I}_a$  and  $\hat{J}_a$  tend respectively to 0 and 1 when  $a$  tends to 0. Integrating on the random variables  $\beta_e$  we deduce that there exists a constant  $C_0$  such that

$$\mathbb{E} \left( \mu_\Lambda^{\eta_{\delta_0, \beta}}(e^{t_x/2}) \right) \leq C_0 \left( \hat{I}_a (\hat{J}_a)^{2d-2} (2d-1) \right)^{|x|},$$

$$\mathbb{E} \left( (\beta_{x,y})^{\frac{1}{4}} \mu_\Lambda^{\eta_{\delta_0, \beta}}(e^{t_x/2}) \right) \leq C_0 \left( \hat{I}_a (\hat{J}_a)^{2d-2} (2d-1) \right)^{|x|},$$

for any  $y \sim x$ . This provides the exponential decrease for all  $a > 0$  such that

$$\hat{I}_a (\hat{J}_a)^{2d-2} (2d-1) < 1.$$

$\square$

*Proof.* (Corollary 2) For any connected finite set  $\Lambda$  containing 0, by Theorems 1 and 2, the ERRW on  $\Lambda$  starting 0 and with constant initial parameter  $a$  is a mixture of reversible Markov chains in conductance  $c_{x,y} = \beta_{x,y} e^{t_x + t_y}$ , where  $\beta_{x,y}$  are  $\text{gamma}(a, 1)$  independent random variables. As in Corollary 1, there exists a constant  $C > 0$  such that

$$\begin{aligned} \mathbb{E} \left( \mu_\Lambda^{\delta_0, \beta} ((c_{x,y}/c_{\delta,0})^{1/4}) \right) &\leq C \mathbb{E} \left( (\beta_{x,y})^{\frac{1}{4}} \left[ \mu_\Lambda^{\delta_0, \beta}(e^{t_x/2}) \mu_\Lambda^{\delta_0/2, \beta}(e^{t_y/2}) \right]^{1/2} \right) \\ &\leq C \mathbb{E} \left( (\beta_{x,y})^{\frac{1}{4}} \mu_\Lambda^{\delta_0, \beta}(e^{t_x/2}) \right)^{1/2} \mathbb{E} \left( (\beta_{x,y})^{\frac{1}{4}} \mu_\Lambda^{\delta_0/2, \beta}(e^{t_y/2}) \right)^{1/2} \end{aligned}$$

The rest of the proof is similar to the proof of Corollary 1.  $\square$

**Acknowledgment.** We are particularly grateful to Krzysztof Gawedzki for a helpful discussion on the hyperbolic sigma model, and for pointing out reference [15]. We would also like to thank Denis Perrot and Thomas Strobl for suggesting a possible link between the limit measure of VRJP and sigma models. We are also grateful to Margherita Disertori for a useful discussion on localization results on the hyperbolic sigma model.

## REFERENCES

- [1] S. Bacallado, J. D. Chodera, and V. Pande. Bayesian comparison of markov models of molecular dynamics with detailed balance constraint. *J. Chem. Phys.*, 131(2):045106, 2009.
- [2] Sergio Bacallado. Bayesian analysis of variable-order, reversible Markov chains. *Ann. Statist.*, 39(2):838–864, 2011.
- [3] A-L. Basdevant and A. Singh. Continuous time vertex reinforced jump processes on Galton-Watson trees. *Preprint, available on <http://arxiv.org/abs/1005.3607>*, 2010.
- [4] A. Collecchio. Limit theorems for Diaconis walk on certain trees. *Probab. Theory Relat. Fields*, 136(1):81–101, 2006.
- [5] Andrea Collecchio. On the transience of processes defined on Galton-Watson trees. *Ann. Probab.*, 34(3):870–878, 2006.
- [6] Andrea Collecchio. Limit theorems for vertex-reinforced jump processes on regular trees. *Electron. J. Probab.*, 14:no. 66, 1936–1962, 2009.
- [7] D. Coppersmith and P. Diaconis. Random walks with reinforcement. *Unpublished manuscript*, 1986.
- [8] B. Davis. Reinforced random walk. *Probab. Theory Relat. Fields*, 84(2):203–229, 1990.
- [9] Burgess Davis and Stanislav Volkov. Continuous time vertex-reinforced jump processes. *Probab. Theory Related Fields*, 123(2):281–300, 2002.
- [10] Burgess Davis and Stanislav Volkov. Vertex-reinforced jump processes on trees and finite graphs. *Probab. Theory Related Fields*, 128(1):42–62, 2004.
- [11] P. Diaconis. Recent progress on de Finetti’s notions of exchangeability. *Bayesian Statistics*, 3 (Valencia,1987), Oxford Sci. Publ., Oxford University Press, New York:111–125, 1988.
- [12] P. Diaconis and D. Freedman. de Finetti’s theorem for Markov chains. *Ann. Probab.*, 8(1):115–130, 1980.
- [13] P. Diaconis and S.W.W. Rolles. Bayesian analysis for reversible Markov chains. *To appear in The Annals of Statistics*, 34(3), 2006.
- [14] M. Disertori and T. Spencer. Anderson localization for a supersymmetric sigma model. *Comm. Math. Phys.*, 300(3):659–671, 2010.
- [15] M. Disertori, T. Spencer, and M. R. Zirnbauer. Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model. *Comm. Math. Phys.*, 300(2):435–486, 2010.
- [16] M.S. Keane and S.W.W. Rolles. Edge-reinforced random walk on finite graphs. *Infinite dimensional stochastic analysis (Amsterdam, 1999) R. Neth. Acad. Arts. Sci*, pages 217–234, 2000.
- [17] M.S. Keane and S.W.W. Rolles. Tubular recurrence. *Acta Math. Hungar.*, 97(3):207–221, 2002.
- [18] David G. Kendall. Branching processes since 1873. *J. London Math. Soc.*, 41:385–406. (1 plate), 1966.
- [19] V. Limic. Attracting edge property for a class of reinforced random walks. *Annals of Probability*, 31:1615–1654, 2003.
- [20] V. Limic and P. Tarrès. Attracting edge and edge reinforced walks. *Annals of Probability*, 35, No. 5:1783–1806, 2007.
- [21] F. Merkl and S. W. W. Rolles. A random environment for linearly edge-reinforced random walks on infinite graphs. *Probab. Theory Relat. Fields*, 138:157–176, 2007.
- [22] F. Merkl and S.W.W. Rolles. Edge-reinforced random walk on one-dimensional periodic graphs. *Probability Theory and Related Fields*, 145:323–349, 2009.
- [23] F. Merkl and S.W.W. Rolles. Recurrence of edge-reinforced random walk on a two-dimensional graph. *To appear in the Annals of Probability*, 2009.
- [24] Franz Merkl and Silke W. W. Rolles. Recurrence of edge-reinforced random walk on a two-dimensional graph. *Ann. Probab.*, 37(5):1679–1714, 2009.
- [25] R. Pemantle. Phase transition in reinforced random walk and RWRE on trees. *Annals of Probability*, 16(3):1229–1241, 1988.
- [26] Robin Pemantle. A survey of random processes with reinforcement. *Probab. Surv.*, 4:1–79 (electronic), 2007.
- [27] S.W.W. Rolles. How edge-reinforced random walk arises naturally. *Probab. Theory Relat. Fields*, 126(2):243–260, 2003.
- [28] S.W.W. Rolles. On the recurrence of edge-reinforced random walk on  $\mathbb{Z} \times G$ . *Probab. Theory and Relat. Fields*, 135(2):216–264, 2006.

- [29] C. Sabot. Random walks in random Dirichlet environment are transient in dimension  $d \geq 3$ . *Probab. Theory Related Fields*, 151(1-2):297–317, 2011.
- [30] C. Sabot. Random dirichlet environment viewed from the particle in dimension  $d \geq 3$ . *To appear in Annals of Probability*, 2012.
- [31] T. Sellke. Reinforced random walks on the  $d$ -dimensional integer lattice. *Technical report 94-26, Purdue University*, 1994.
- [32] P. Tarrès. Vertex-reinforced random walk on  $\mathbb{Z}$  eventually gets stuck in five points. *Annals of Probability*, 32, No.3B:2650–2701, 2004.
- [33] P. Tarrès. Localization of reinforced random walks. *Available on <http://arxiv.org/abs/1103.5536>*, 2011.
- [34] B. Tóth. Generalized Ray-Knight theory and limit theorems for self-interacting random walks on  $\mathbb{Z}$ . *Annals of Probability*, 24:1324–1367, 1996.

UNIVERSITÉ DE LYON, UNIVERSITÉ LYON 1, INSTITUT CAMILLE JORDAN, CNRS UMR 5208, 43, BOULEVARD DU 11 NOVEMBRE 1918, 69622 VILLEURBANNE CEDEX, FRANCE

*E-mail address:* `sabot@math.univ-lyon1.fr`

UNIVERSITÉ PAUL SABATIER, INSTITUT DE MATHÉMATIQUES DE TOULOUSE, CNRS UMR 5219, 118 ROUTE DE NARBONNE, TOULOUSE CEDEX 9, FRANCE. ON LEAVE FROM THE MATHEMATICAL INSTITUTE, UNIVERSITY OF OXFORD.

*E-mail address:* `pierre.tarres@math.univ-toulouse.fr`